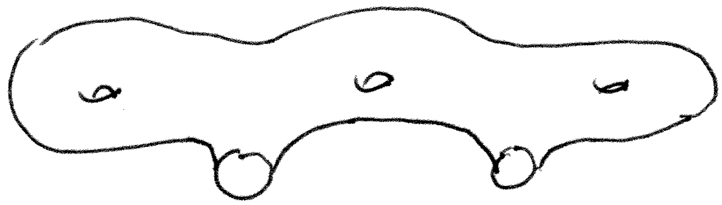
A Trip to the Zoo: The Curve Complex of a Closed Surface

Def: 1) $S_{g,b} :=$ closed surface of genus g with b boundary components.



$S_{3,2}$

2) Curve $\alpha \subset S$ embedded copy of S^1

- seperating if $S - \alpha$ has 2 components.
- essential if no component of $S - \alpha$ has closure being a disk
- non-peripheral if no component of $S - \alpha$ is an annulus.

3) Arc $\alpha \subset S$ proper ~~the~~ ~~embedded~~ embedding of $[0,1]$.

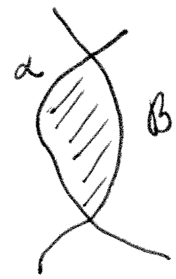
- essential if $S - \alpha$ has no component w/ closure being a disk.

Def: $i(\alpha, \beta) :=$ geometric intersection #

$=$ minimal # of intersections of α', β' if $\alpha \overset{\text{isotopic}}{\sim} \alpha'$ $\beta \overset{\text{isotopic}}{\sim} \beta'$

Rmk/Exer: This happens exactly when $S - (\alpha \cup \beta)$ has no

bigons



(curves)

boundary triangles

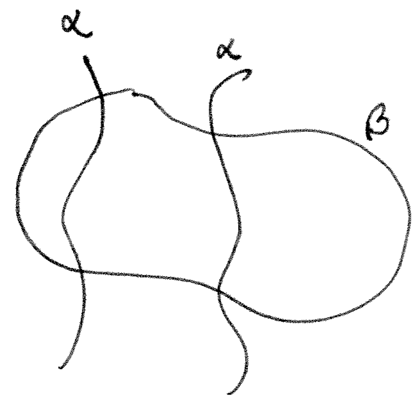
(arcs).

Prop: Suppose α, β meet once, transversely. Then $i(\alpha, \beta) = 1$

Pf: Differential topology
- intersection # is ± 1

Prop: Suppose $i(\alpha, \beta) > 0$. Prove α, β are essential, non-peripheral.

Pf: If they were, could isotope them tiny enough s.t. $i(\alpha, \beta) = 0$



<u>Def</u>: Curve Complex $S_{g,b}$

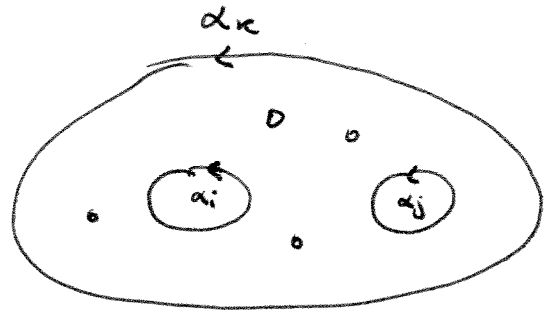
$C(S_{g,b})$: vertices: isotopy classes of essential non-peripheral curves in S.

$\alpha_0 \dots \alpha_k$ form a k-simplex if they can be realized in $S_{g,b}$ disjointly.

<u>Prop</u>: Top dimensional simplices have
$$\xi(S_{g,b}) := 3g-3+b \quad \text{simplices.}$$
"complexity" of the surface

<u>Pf</u>: Recall we know this when $b=0$ (pants decomp).

If you have a maximal simplex, take $\alpha_0 \dots \alpha_{3g-3} \rightsquigarrow$ pants decomp. of $S_{g,b}$.



- b boundary components.

Use induction to show need exactly b curves in this pair of pants.

(Caution: may get annuli or pants when you induct).

Exer: $C(S_{0,5}) \cong C(S_{1,2})$

$C(S_{0,6}) \cong C(S_{0,2})$

$C(S_{0,6}) \not\cong C(S_{1,3})$

Focus on 1-skeleton of $C(S)$ for rest of talk.

Def: $d_S(\alpha,\beta) :=$ distance between α,β in the 1-skeleton of $C(S)$.

Prop (Hempel)
Let S be a ~~connected~~ compact, connected surface. Suppose α,β are curves, $i(\alpha,\beta) \neq 0$. Then

$$d_S(\alpha,\beta) \leq 2\log_2(i(\alpha,\beta)) + 2.$$

Proof: • Check this for pants, annuli, $S_{1,0}$, $S_{1,1}$, $S_{0,4}$

• Verify when $i(\alpha,\beta) \leq 3$

Suppose $i(\alpha,\beta) = n > 3$. Isotope α,β to realize this, orient α.

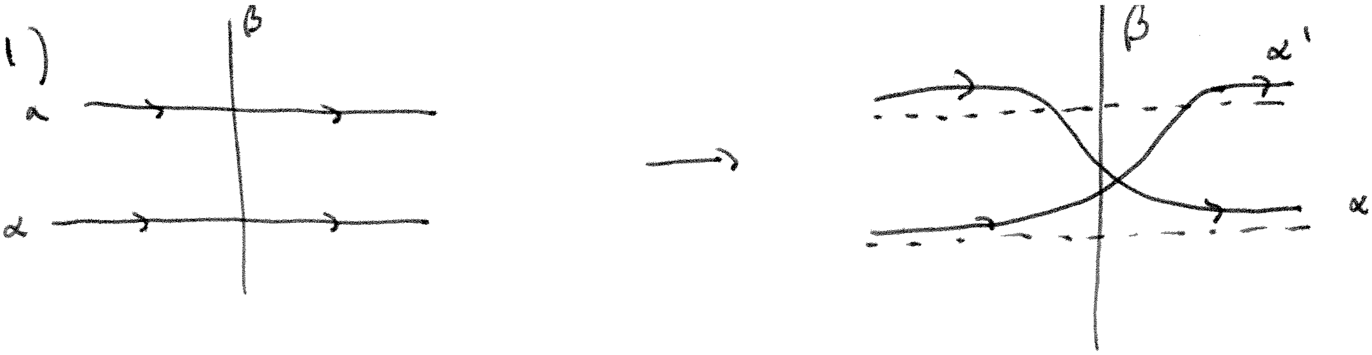
Idea: use induction + curve surgery.

2 cases:

1) 2 consecutive intersections of β meet α in same orientation

2) They don't.

1)



<u>Observe</u>: 1) $i(\alpha,\alpha') = i(\alpha,\alpha'') = i(\alpha',\alpha'')$

$\Rightarrow$ ~~all~~ α, α' essential, non-peripheral (ind.)

& $d_S(\alpha,\alpha') \le 2$

$d_S(\alpha,\alpha'') \le 2$.

2) $i(\alpha',\beta) + i(\alpha'',\beta) \le i(\alpha,\beta)$

Assume w.l.o.g. that $i(\alpha',\beta) \le \frac{1}{2} i(\alpha,\beta)$

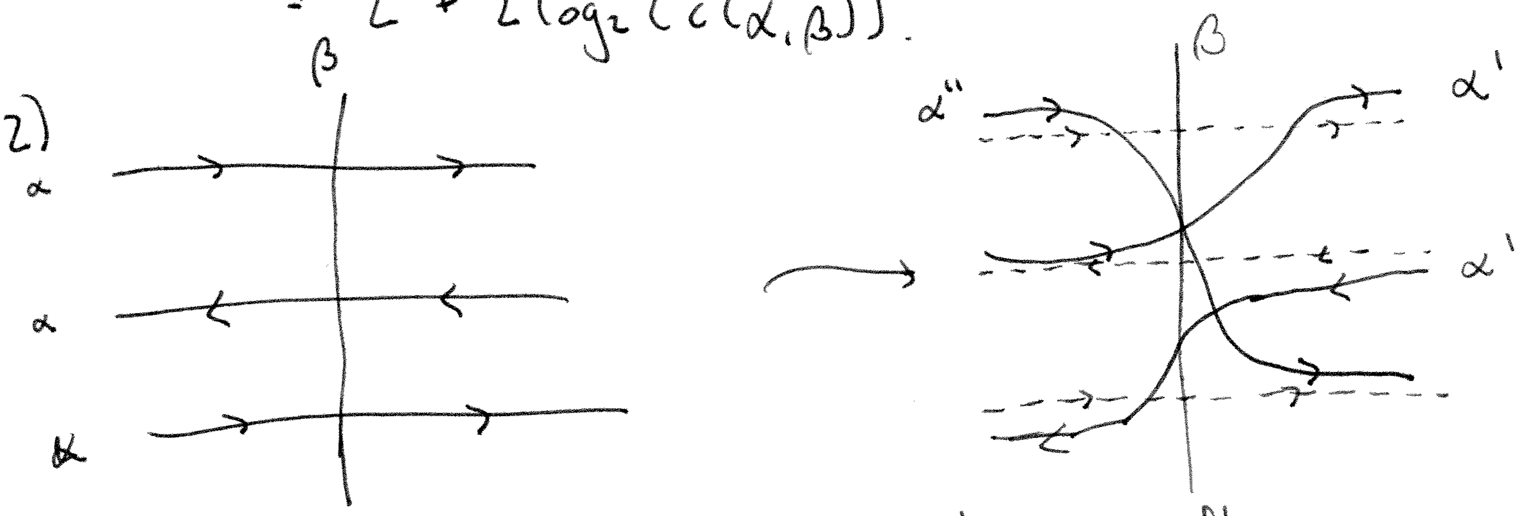
$$\Rightarrow d_S(\alpha,\beta) \le d_S(\alpha,\alpha') + d_S(\alpha',\beta)$$

$$\le 2 + 2 + 2\log_2(i(\alpha',\beta))$$

$$\le 2 + 2 + 2\log_2\left(\tfrac{1}{2} i(\alpha,\beta)\right)$$

$$\le 2 + 2\log_2(i(\alpha,\beta)).$$

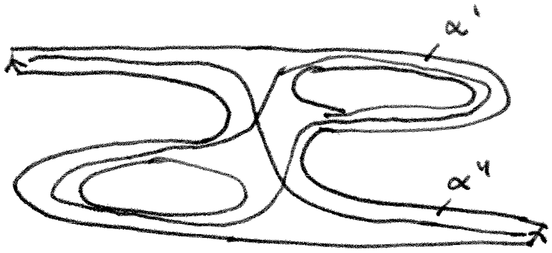
2)



- this or ~~flip~~ reflected vertically

Claim: $d_S(\alpha,\alpha') = d_S(\alpha,\alpha'') \geq 2$.

Pf: Small nbhd of α', α'' looks like $S_{0,4}$.



Since our surface was not $S_{0,4}$
$\Rightarrow$ One of the boundary components is essential & non-peripheral in S,
so it suffices to show $d_S(\alpha,\alpha') = d_S(\alpha,\alpha'') = 2$.

Claim: $c(\alpha,\alpha'') = c(\alpha,\alpha') = c(\alpha,\alpha'') = 2$

≤ 2 clear.

Cannot be 1 b/c intersection # from diff top is $0 \pmod 2$.

Cannot be 0.

$\Rightarrow$ we could isotope α, β to reduce $c(\alpha,\beta)$ ~~intersection number~~, $\lightning$

$\Rightarrow$ α', α'' essential non-peripheral, done by induction.

Rmk: Not sharp. (extremely.)

~~$f \in MCG(S)$ $f \curvearrowright S$ $\rightsquigarrow$~~

$MCG(S) \curvearrowright S \rightsquigarrow MCG(S) \curvearrowright \mathcal{C}(S)$.

Def: $f \in MCG(S)$, $f \curvearrowright \mathcal{C}(S)$

- elliptic - every orbit of f is bounded.
- hyperbolic - every orbit ~~is a~~ of f is a quasi-isometric embedding of $\mathbb{Z}$
- Parabolic - if neither.

Thm (Masur Minsky)

- Periodic + reducible elements of $MCG(S)$ act elliptically on $\mathcal{C}(S)$
- Pseudo-Anosov elements act hyperbolically
  (Dehn twists act hyperbolically on $\mathcal{C}(S_{0,2})$)
  * maybe leave this out

Cor: $\mathcal{C}(S)$ has infinite diameter

Thm (Masur-Minsky)
The curve complex is hyperbolic.

Thm (Aougab, Hensel, Przytycki + Webb)
(Clay, Rafi, ~~Sh~~ Schleimer)

$\exists k$ s.t. $C(S_{g,p})$ is k-hyperbolic for all g,p s.t. $3g+p \geq 5$
(1-skeleton)

Prop Given $h \geq 0$, $k \geq 0$ s.t.
G connected ~~subgraph~~ graph s.t.
$x,y \in V(G)$, $\mathcal{L}(x,y) \subseteq G$ subgraph with
$x,y \in \mathcal{L}(x,y)$ and

1) $\mathcal{L}(x,y) \subseteq N(\mathcal{L}(x,z) \cup \mathcal{L}(z,y), h)$

2) If $d(x,y) \leq 1$, diameter of $\mathcal{L}(x,y)$ is $\leq h$.

Then G is k-hyperbolic where

$$k \geq (3m - 10h)/2$$

with m s.t. $2h(6 + \log_2(m+2)) \leq m$.

Apply this cleverly to the correct sets
α, β.